

Impact Dynamics Lab

Choosing model fidelity for a pendulum with a mechanical stop

Independent simulation project · 2026

This project examines a practical modeling decision: how much detail is needed to predict a mechanism reaching its travel limit? An interactive browser demonstrator links the motion of a pendulum bob, its phase trajectory, collision events, and energy loss. Live controls expose how stop angle and restitution change the response.

A physical mechanism behind the equations

A ball on a light rigid rod swings into an adjustable stop. It is a simple, buildable analogue for a passive linkage approaching an end stop. The quantities of interest—collision time, incoming speed, rebound amplitude, and impulse—connect the model to timing, clearance, and momentum-transfer decisions. Pendulum dynamics also form building blocks of articulated mechanisms. [1]

The stop is radial: its face is parallel to the rod at contact, so the stop normal is tangent to the bob's circular path. The displayed ball radius is schematic. The model assumes a point mass, massless rigid rod, frictionless pivot, rigid stop, instantaneous collision, and constant restitution. It is not a calibrated model of a manufactured device.

Two flight models, one impact law

With $\tau = \sqrt{g/L} t$, $v = d\theta/d\tau$, and stop angle α ,

$$\begin{aligned} \text{Full sine flight:} & \quad \theta'' + \sin \theta = 0, \\ \text{Linearized flight:} & \quad \theta'' + \theta = 0, \\ \text{Incoming contact at } \theta = \alpha : & \quad v^+ = -e v^-, \quad v^- < 0. \end{aligned}$$

Both complete simulations are *hybrid systems*: continuous flight is interrupted by a state-triggered velocity reset. Only the approximated flight equation is linear. Position stays continuous while velocity jumps; the impact removes $\frac{1}{2}(1 - e^2)(v^-)^2$ of dimensionless energy.

A reproducible approximation-error study

Release from rest; vertical stop $\alpha = 0$; $e = 0.60$. Percentages compare the linearized prediction with the full sine prediction, which is a mathematical reference rather than measured ground truth.

Release angle	Free period too short	First contact time too early	Incoming speed too high
10°	0.19%	0.19%	0.13%
30°	1.71%	1.71%	1.15%
60°	6.82%	6.82%	4.72%
90°	15.28%	15.28%	11.07%

Underprediction is $100(N - H)/N$; speed overprediction is $100(H - N)/N$, where H and N denote the respective predictions. Free periods are evaluated without a stop. For this vertical-stop experiment, first contact occurs after one quarter of the free period. An illustrative 1% timing target is met at 10° and missed at 30°.

Why a linear model can be the better engineering choice

A linear approximation is useful near the operating point for which it was derived. [2] When its error is small enough, the simpler model can answer the design question with less analytical and interpretive effort. Superposition, local stability, transfer functions, frequency response, and established controller-design methods make linear models especially useful around a nominal condition.

The result is not a claim that linear models are generally more accurate, or that nonlinear models are unusual. Nonlinear and contact models are established engineering tools. [3,4] The choice depends on the required accuracy, operating range, parameters that can be measured, and the consequence of an incorrect prediction. Added detail can increase calibration and validation work without changing the decision.

When the nonlinear flight model changes the decision

At large angles, replacing $\sin \theta$ by θ overstates the restoring torque. That changes both arrival time and collision speed. Consider a 90° release, a 15° stop and $e = 0.95$, with *assumed* $L = 0.50$ m, $m = 0.050$ kg and $g = 9.81$ m/s²:

Predicted quantity	Linearized flight	Full sine flight
First contact time	0.3168 s	0.3765 s
Incoming normal speed	3.430 m/s	3.078 m/s
Normal impulse	0.3344 N s	0.3001 N s

A roughly 60 ms collision-time shift could matter to an event-timed mechanism. For this radial-stop geometry,

$$u_n = \sqrt{gL} |v^-|, \quad J_n = m\sqrt{gL} (v^+ - v^-).$$

Impulse is a momentum transfer, not a peak force. Peak contact force requires a compliant-contact model or measured force history; an instantaneous velocity reset does not identify it.

Both flight models reproduce the three qualitative stop-angle regimes. A stop before vertical supports the bob after a finite accumulation of shrinking rebounds; a vertical stop gives asymptotic decay; a stop beyond vertical permits a limiting grazing oscillation. The full sine term improves quantitative predictions. It is not the reason impacts or these regimes exist.

Computation cost measured on this implementation

After warm-up, 31 alternating solves per model gave median/p95 times of 0.698/1.219 ms for harmonic flight and 12.221/16.726 ms for full sine flight. These are MATLAB R2026b solve times on the development machine, including sampled arrays and impact traces but excluding graphics. The benchmark uses a 90° release, 15° stop, $e = 0.95$, $\tau \in [0, 18]$, and 0.01 sample spacing.

This solver exploits analytic harmonic and elliptic flight solutions. The measured difference is specific to these implementations and is not a universal cost ratio for nonlinear modeling. Both are inexpensive enough for interactive exploration; rendering latency is a separate measurement.

Verification that supports the project

The core has 15 numerical validation groups covering conservation during flight, restitution loss, elastic impacts, nonpenetration, positive/zero/negative stop angles, contact accumulation, and grazing limits. A separate `ode45` implementation with event detection agrees with the tested full sine collision times to 9.1×10^{-12} in τ , and interior states to about 10^{-11} .

Those differences establish numerical consistency within the tested model, not physical accuracy. The unresolved tail of rapid rebounds has an explicit bound before continuation as supported contact. A real rig has finite contact compliance, losses, and measurement resolution that regularize the ideal accumulation.

Interactive exploration and reproducibility

Open Impact Dynamics Lab in a modern browser. Change the stop angle, release angle, or restitution to recompute the same initial-condition experiment at the displayed time. Motion and phase plots use the same computed states. Changes preserve play/pause and time; they do not model a moving wall. The browser edition assumes $L = 0.50$ m and $g = 9.81$ m/s², displays seconds and rad/s, and exports both flight models in a CSV.

The portable source archive contains the numerical module, interface, and tests. With Node.js 20 or later, run `node -test tests/physics.test.mjs`. The browser suite passed 45 tests on October 18, 2026, including 24 frozen MATLAB reference cases, 21,624 sampled states, and 347 near-impact probes. Read the included verification summary for errors and model limitations. These checks supplement the MATLAB-edition results above; browser runtimes are not the benchmark timings reported above.

The full sine solver covers librations below the separatrix; continuous rotation is outside this implementation.

A proposed bench test

Measure rod length, bob mass, stop geometry and their uncertainty. Use free-swing trials to characterize pivot and air losses. Estimate restitution from angular speeds immediately before and after distinct impacts, then predict contact time and rebound amplitude for held-out release and stop angles. Report repeatability and measurement uncertainty, not just visual agreement.

Only add speed-dependent restitution, rod inertia, or contact compliance when measurements show that they matter to the required accuracy. A force sensor and known measurement bandwidth would be needed to validate a contact-force history. This test plan has not been performed; no prototype or measured results are claimed.

New regime and energy study

A dependency-free browser-model study now records 50 combinations: both flight models, stop angles $-30, -15, 0, 15, 30$ degrees, and restitution $0.2, 0.6, 0.9, 0.99, 1$, released from rest at 60 degrees over $0 \leq \tau \leq 120$. The study closes the energy budget

$$E(0) - \sum_k \frac{1}{2}(1 - e^2)(v_k^-)^2 - E(120)$$

within 4.94×10^{-11} ; sampled stop penetration is zero. The event ledger excludes the unresolved microscopic impacts represented by the explicitly bounded positive-stop tail. These are bounded numerical checks, not a proof for every possible input.

For the full-sine model at a 15° stop, the computed contact time rises from 3.0803τ at $e = 0.2$ to 9.3825τ at $e = 0.6$ and 45.2938τ at $e = 0.9$. At $e = 0.99$, contact is bounded beyond the 120τ study window; its exact time is not reported. At $e = 1$, no energy is dissipated and no finite dissipative contact occurs. Zero and negative stops retain their asymptotic regimes rather than being clamped to rest.

Twelve tail-tolerance cases use $10^{-6}, 10^{-8}, 10^{-10}$ in τ for both flight models at $e = 0.6, 0.9$. The largest contact-time change is $2.22 \times 10^{-8}\tau$, within the summed reported bounds plus the 10^{-10} comparison allowance. A separately summed harmonic flight-time series checks contact independently. The four-angle approximation table is regenerated from the same browser core.

Reproduce and inspect the extension

The source archive includes `study/config.json`, `study/regime-study.mjs`, and the resulting CSV, JSON and SVG files. Run `node study/regime-study.mjs` to regenerate them and `node study/validate.mjs` to run the numerical suite and refresh reference metrics and source hashes. No third-party numerical library is required.

The sampled-run CSV contains the displayed states. A separate exact-impact CSV retains incoming and outgoing angular velocities, energy loss and normal impulse, with the assumed $m = 0.050$ kg, $L = 0.50$ m and $g = 9.81$ m/s² in every row. Physical energy is $mgLE$; impulse is $m\sqrt{gL}(v^+ - v^-)$. No peak contact force is inferred. Changing length scales time as $\sqrt{L/g}$ and angular speed as $\sqrt{g/L}$; it does not change the dimensionless trajectory.

Project provenance

This 2026 independent redevelopment extends a 2021 pendulum-impact model. The new scope comprises event-resolved dynamics, live controls, numerical checks, model comparison, reproducible benchmarks, and this engineering case study. Historical source files remain preserved; the source review records their provenance.

References

- [1] R. Tedrake, [Underactuated Robotics: The Simple Pendulum](#).
- [2] MathWorks, [Linearization](#).
- [3] MathWorks, [Rotational Hard Stop](#).
- [4] R. Tedrake, [Planning and Control through Contact](#).

Project evidence: benchmark script, comparison data, and numerical tests. References accessed 17 October 2026.